49. The base of a solid is bounded by $y = x^3$, $y = 0$, and $x = 1$. Find the volume of the solid if the cross sections perpendicular to the y-axis are
(a) squares
(b) semicircles
(c) equilateral triangles
(d) trapezoids for which $h = b_1 = \frac{1}{2}b_2$, where b_1 and b_2 are the lengths of the upper and lower bases
(e) semiellipses whose heights are twice the lengths of their bases

50. Find the volume of the solid of intersection (the solid common to both) of the two right circular cylinders of radius r whose axes meet at right angles, as shown in Figure 7.28.

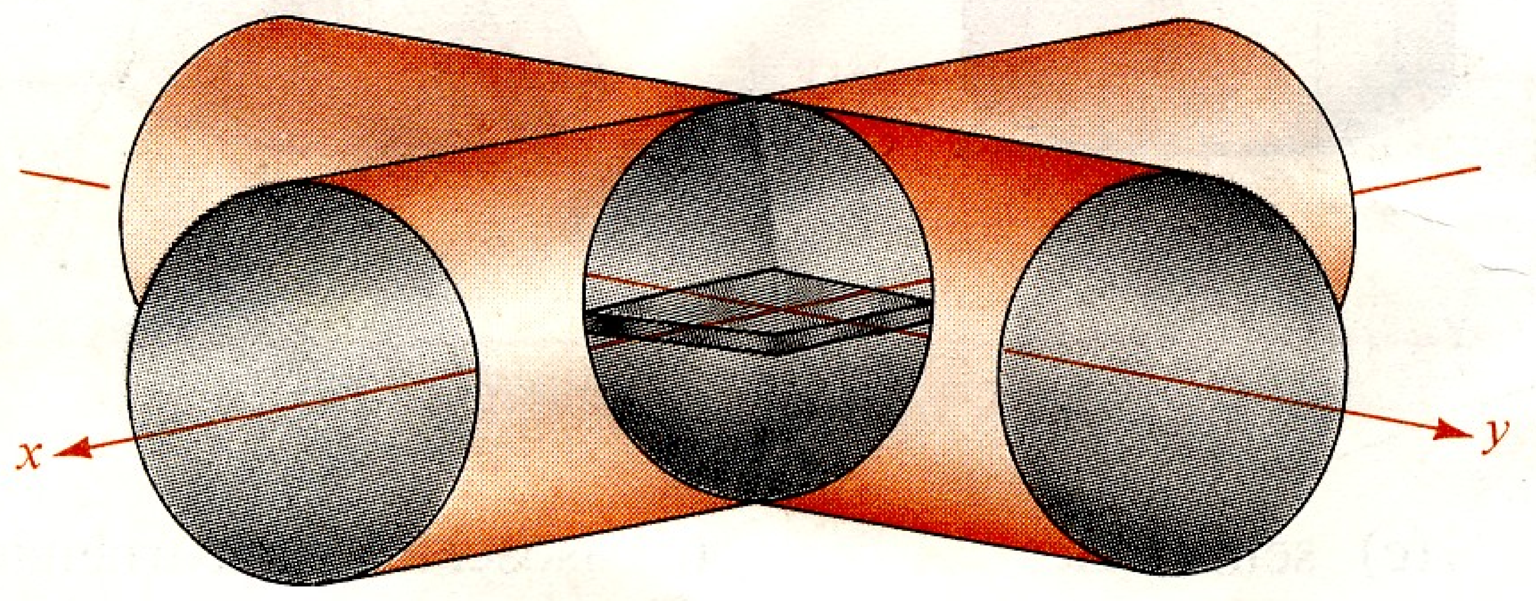


FIGURE 7.28

51. A wedge is cut from a right circular cylinder of radius r inches by a plane through the diameter of the base, making an angle of 45° with the plane of the base, as shown in Figure 7.29. Find the volume of the wedge.

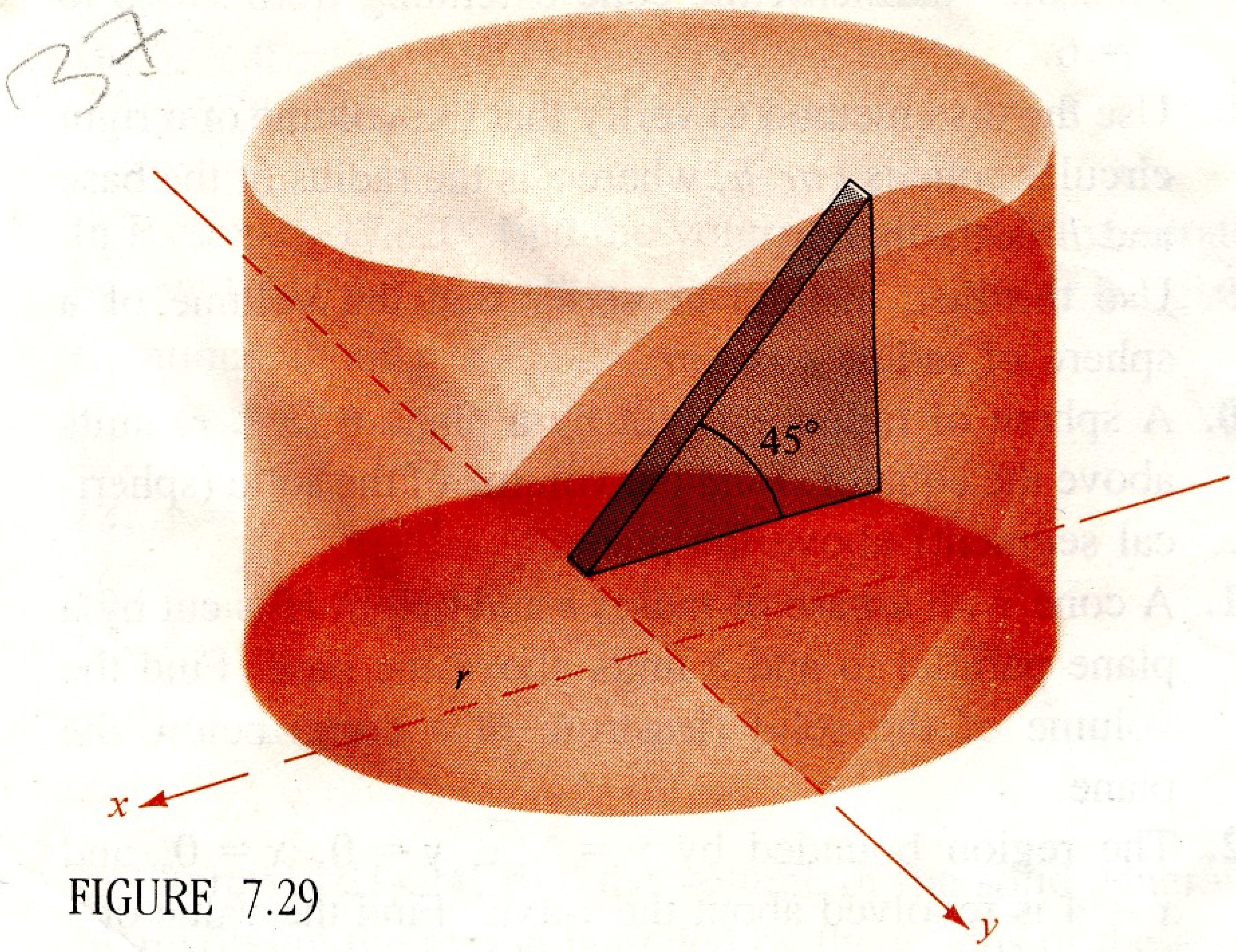


FIGURE 7.29

7.3 Volume: The shell method

SECTION TOPICS ▪
The shell method ▪
Comparison of disc and shell methods ▪

In this section, we look at an alternative method for finding the volume of a solid of revolution, a method that uses cylindrical shells. Both the disc method and the shell method are important and we will compare the advantages of one method over the other later in this section.

To introduce the **shell method,** we consider a representative rectangle as shown in Figure 7.30, where

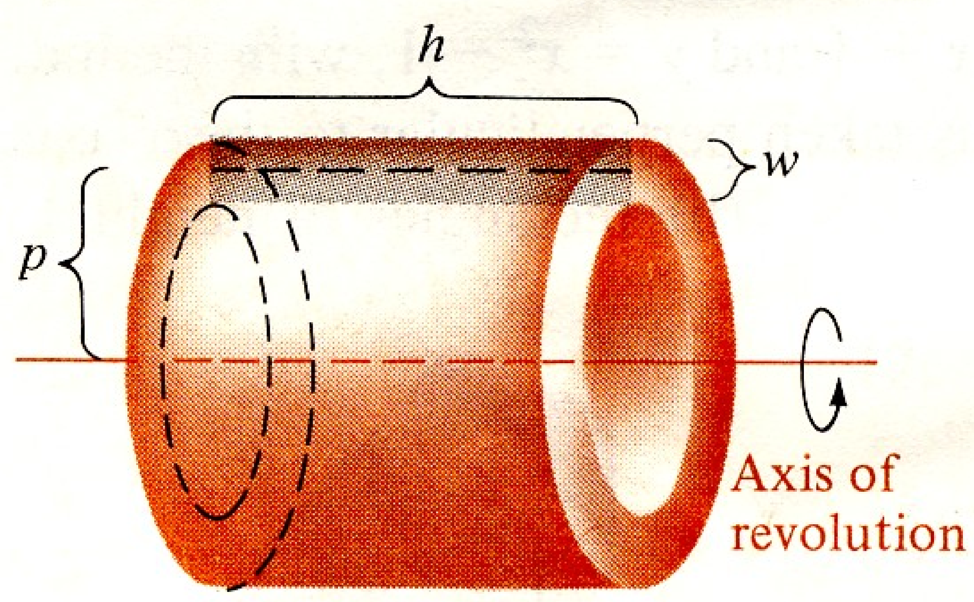


FIGURE 7.30

w = width of the rectangle

h = height of the rectangle

p = distance between axis of revolution and *center* of the rectangle

When this rectangle is revolved about its axis of revolution, it forms a cylindrical shell (or tube) of thickness w. To find the volume of this shell we consider two cylinders. The radius of the larger cylinder corresponds to the outer radius of the shell and the radius of the smaller cylinder corresponds to the inner radius of the shell. Since p is the average radius of the shell, we know the outer radius is $p + (w/2)$ and the inner radius is $p - (w/2)$. Thus, the volume of the shell is given by the difference:

$$\begin{aligned}\text{volume of shell} &= (\text{volume of cylinder}) - (\text{volume of hole})\\ &= \pi\left(p + \frac{w}{2}\right)^2 h - \pi\left(p - \frac{w}{2}\right)^2 h\\ &= \pi\left(p^2 + pw + \frac{w^2}{4} - p^2 + pw - \frac{w^2}{4}\right)h\\ &= \pi 2phw = 2\pi(\text{average radius})(\text{height})(\text{thickness})\end{aligned}$$

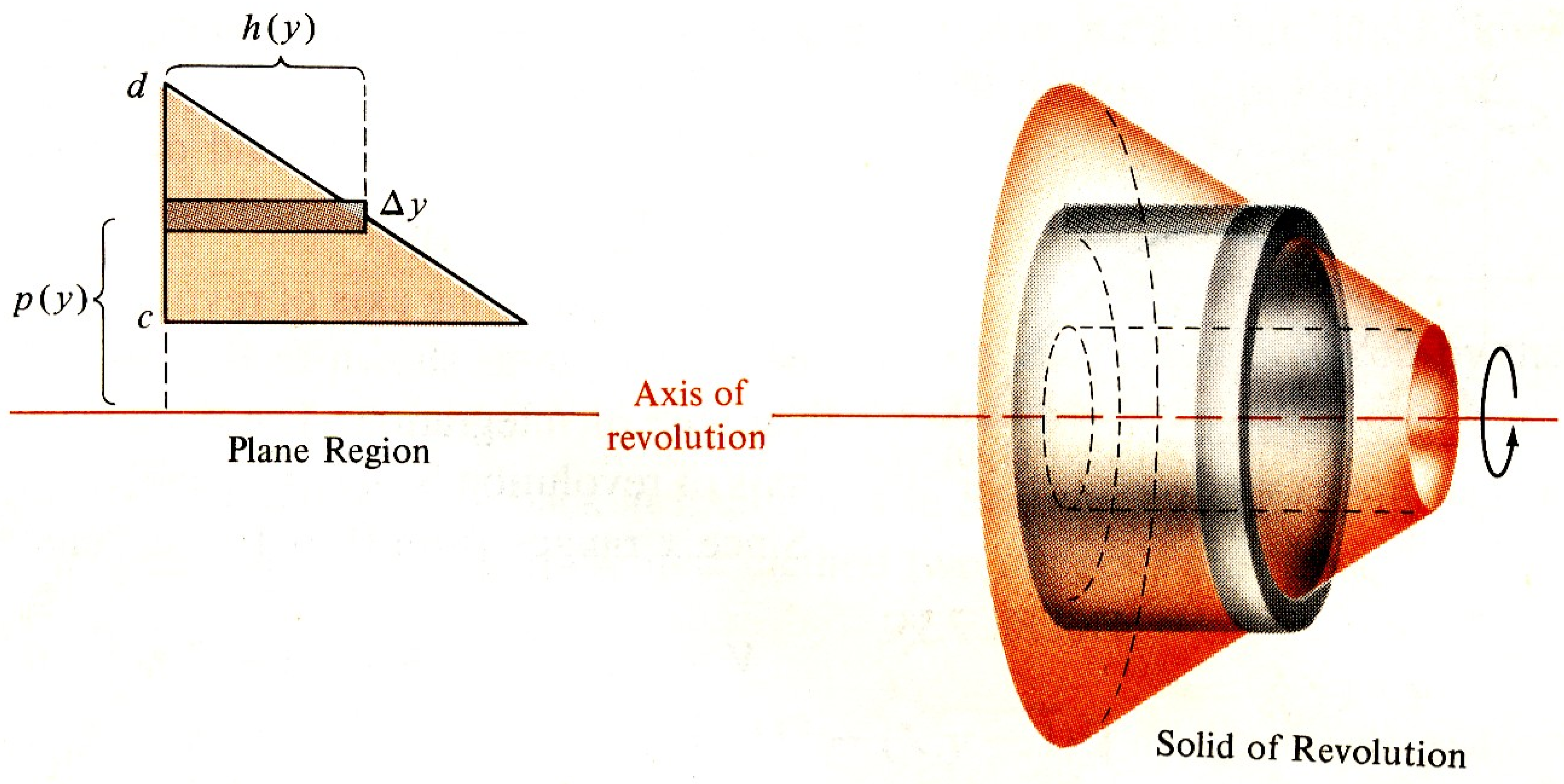


FIGURE 7.31

We use this formula to find the volume of a solid of revolution as follows. Assume that the plane region in Figure 7.31 is revolved about a line to form the indicated solid. If we consider a horizontal rectangle of width Δy, then as the plane region is revolved about its axis of revolution, the rectangle generates a representative shell whose volume is

$$\Delta V = 2\pi[p(y)h(y)]\ \Delta y$$

Now, if we approximate the volume of the solid by n such shells of thickness Δy, height $h(y_i)$, and average radius $p(y_i)$, we have

$$\text{volume of solid} \approx \sum_{i=1}^{n} 2\pi[p(y_i)h(y_i)]\ \Delta y = 2\pi \sum_{i=1}^{n} [p(y_i)h(y_i)]\ \Delta y$$

By taking the limit as $||\Delta|| \to 0$ $(n \to \infty)$, we have

$$\text{volume of solid} = \lim_{n\to\infty} 2\pi \sum_{i=1}^{n} [p(y_i)h(y_i)]\ \Delta y = 2\pi \int_c^d [p(y)h(y)]\ dy$$

THE SHELL METHOD To find the volume of a solid of revolution with the **shell method,** use one of the following, as shown in Figure 7.32.

Horizontal axis of revolution

$$\text{volume} = V = 2\pi \int_c^d p(y)h(y)\ dy$$

Vertical axis of revolution

$$\text{volume} = V = 2\pi \int_a^b p(x)h(x)\ dx$$

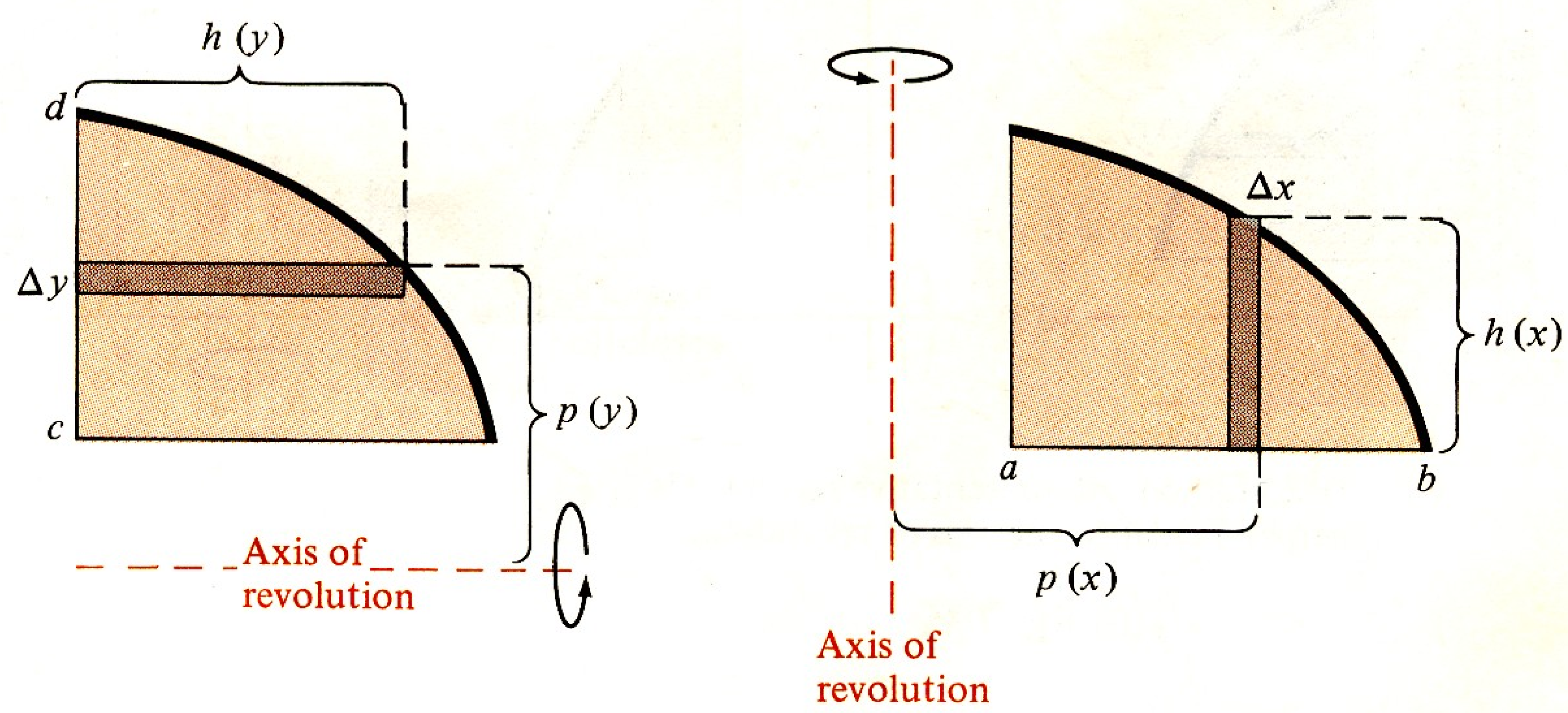


FIGURE 7.32

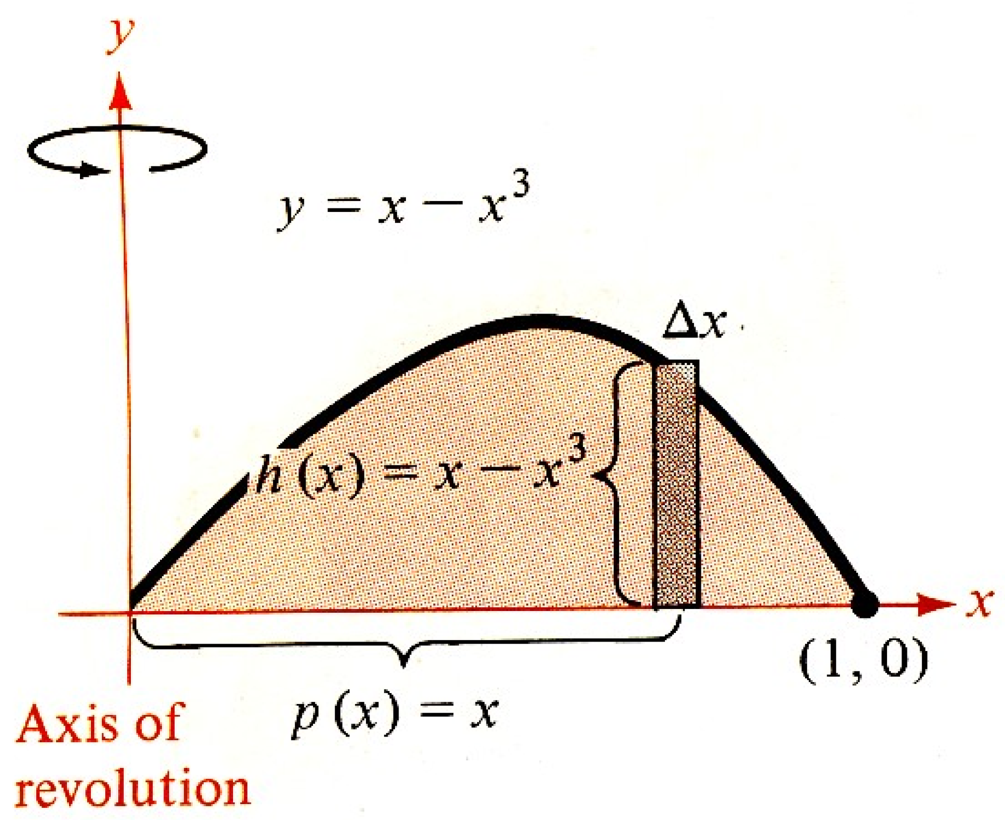


FIGURE 7.33

EXAMPLE 1 Using the shell method to find volume

Find the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$ and the x-axis $(0 \le x \le 1)$ about the y-axis.

Solution: Since the axis of revolution is vertical, we use a vertical representative rectangle as shown in Figure 7.33. The width Δx indicates that x is the variable of integration. The distance from the center of the rectangle to the axis of revolution is $p(x) = x$ and the height of the rectangle is $h(x) = x - x^3$. Since x ranges from 0 to 1, the volume of the solid is

$$V = 2\pi \int_a^b p(x)h(x)\,dx = 2\pi \int_0^1 x(x - x^3)\,dx = 2\pi \int_0^1 (-x^4 + x^2)\,dx$$

$$= 2\pi\left[-\frac{x^5}{5} + \frac{x^3}{3}\right]_0^1 = \frac{4\pi}{15}$$

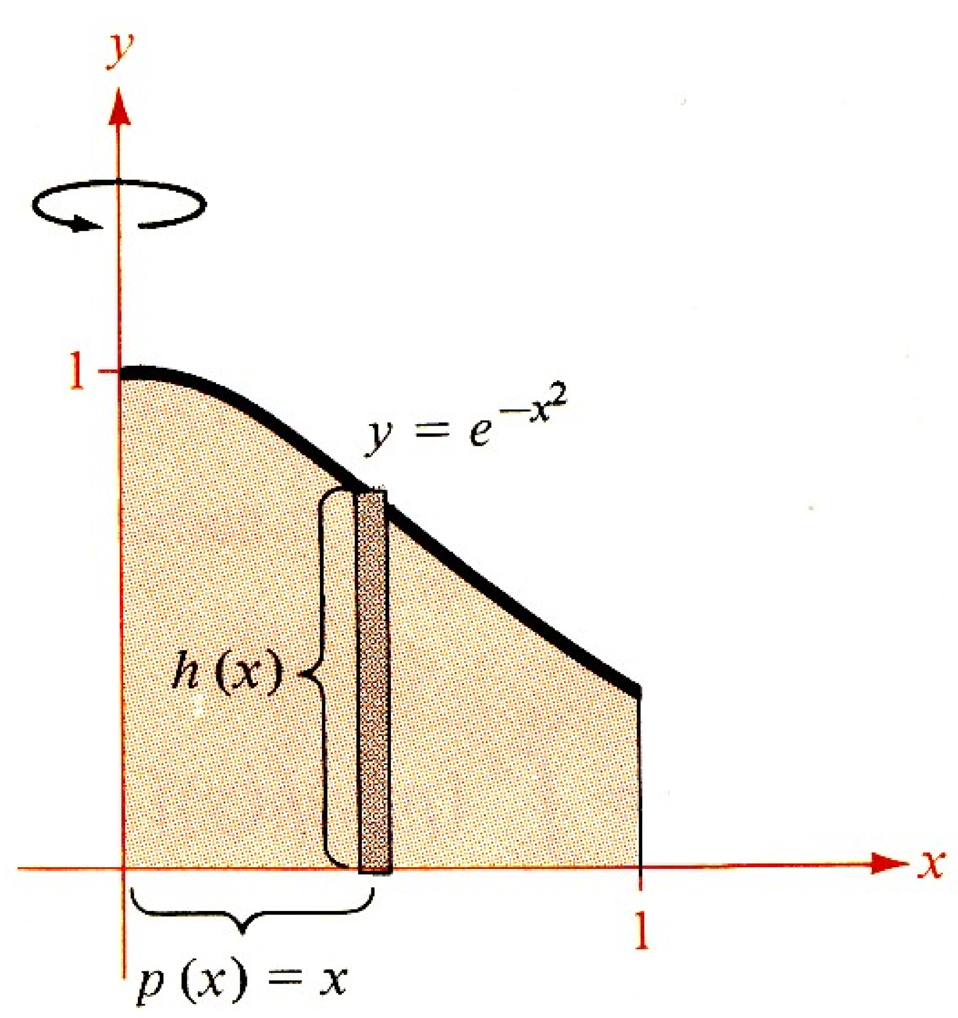


FIGURE 7.34

EXAMPLE 2 Using the shell method to find volume

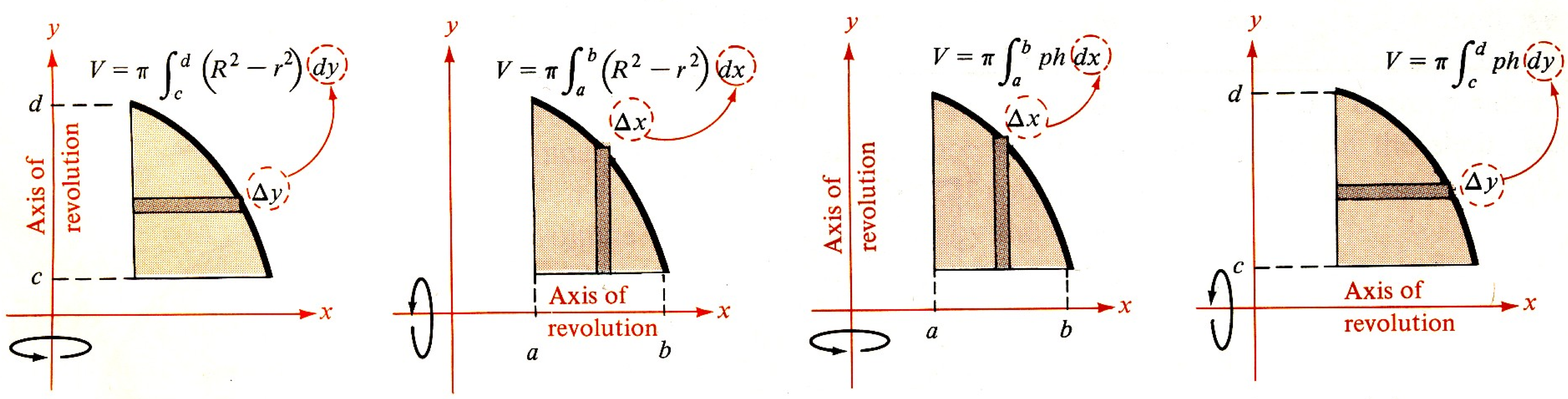
Find the volume of the solid of revolution formed by revolving the region bounded by $y = e^{-x^2}$ and the x-axis $(0 \le x \le 1)$ about the y-axis.

Solution: Since the axis of revolution is vertical, we use a vertical representative rectangle as shown in Figure 7.34. The width Δx indicates that x is the variable of integration. The distance from the center of the rectangle to the axis of revolution is $p(x) = x$ and the height of the rectangle is $h(x) = e^{-x^2}$. Since x ranges from 0 to 1, the volume of the solid is

$$V = 2\pi \int_a^b p(x)h(x)\,dx = 2\pi \int_0^1 xe^{-x^2}\,dx = \left[-\pi\, e^{-x^2}\right]_0^1$$

$$= \pi\left(1 - \frac{1}{e}\right) \approx 1.986$$

Remark The disc and shell methods can be distinguished as follows. For the disc method the representative rectangle is always *perpendicular* to the axis of revolution, whereas for the shell method the representative rectangle is always *parallel* to the axis of revolution, as shown in Figure 7.35.



Disc Method: Representative rectangle is perpendicular to the axis of revolution.

Shell Method: Representative rectangle is parallel to the axis of revolution.

FIGURE 7.35

Often, one method is more convenient to use than the other. The following example illustrates a case in which the shell method is preferable.

EXAMPLE 3 Shell method preferable

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.

Solution: To begin, look back at Example 4 in Section 7.2. There, in Figure 7.21, you can see that with the disc method two integrals are needed to find the volume.

$$V = \pi \int_0^1 (1^2 - 0^2)\, dy + \pi \int_1^2 [1^2 - (\sqrt{y-1})^2]\, dy \qquad \text{Disc method}$$

$$= \pi \int_0^1 1\, dy + \pi \int_1^2 (2 - y)\, dy$$

$$= \pi y \Big]_0^1 + \pi \left[2y - \frac{y^2}{2}\right]_1^2$$

$$= \pi\left(1 + 4 - 2 - 2 + \frac{1}{2}\right) = \frac{3}{2}\pi$$

From Figure 7.36 we can see that the shell method requires only one integral to find the volume.

$$V = 2\pi \int_a^b p(x)h(x)\, dx \qquad \text{Shell method}$$

$$= 2\pi \int_0^1 x(x^2 + 1)\, dx$$

$$= 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^1$$

$$= 2\pi\left(\frac{3}{4}\right) = \frac{3\pi}{2}$$

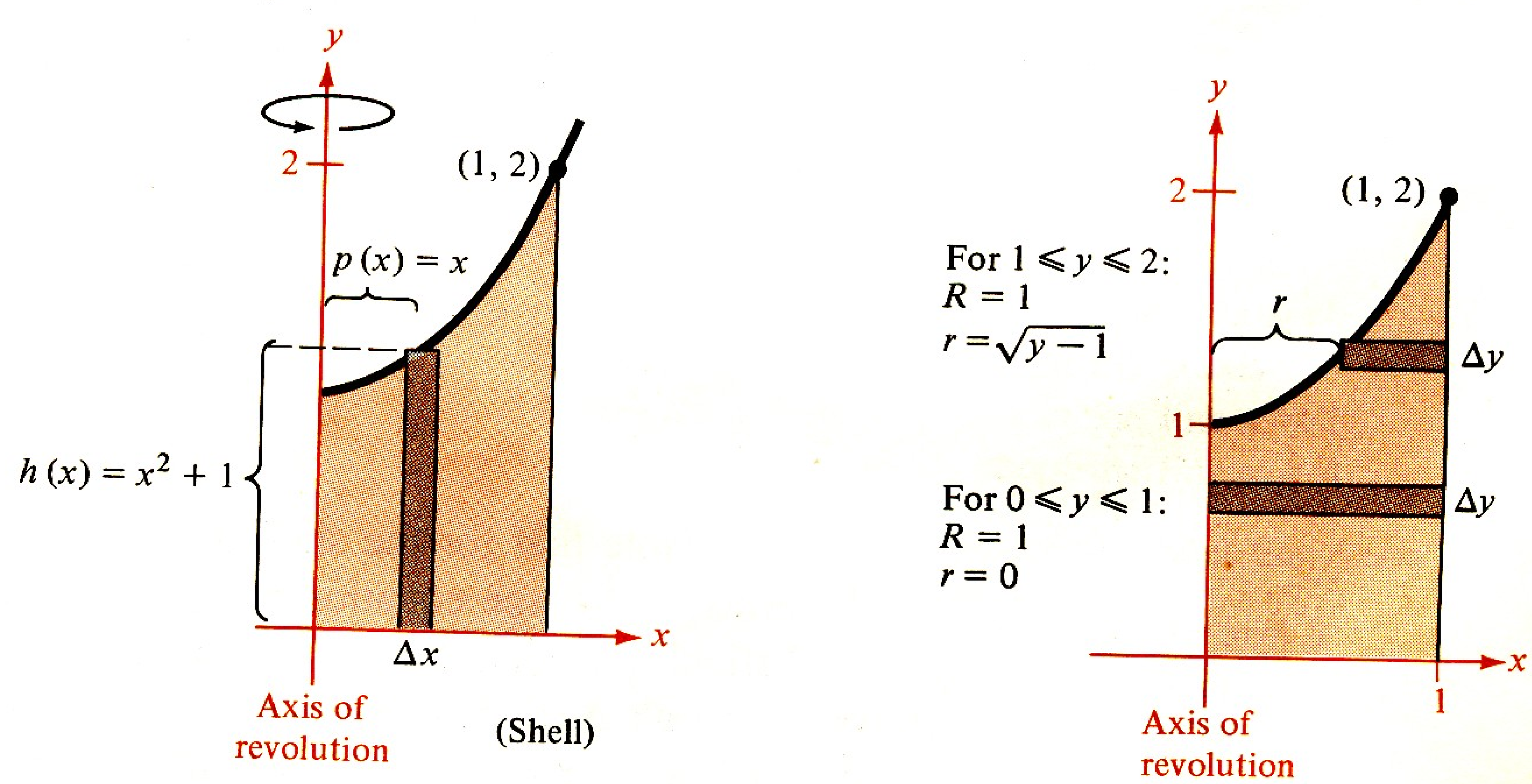


FIGURE 7.36

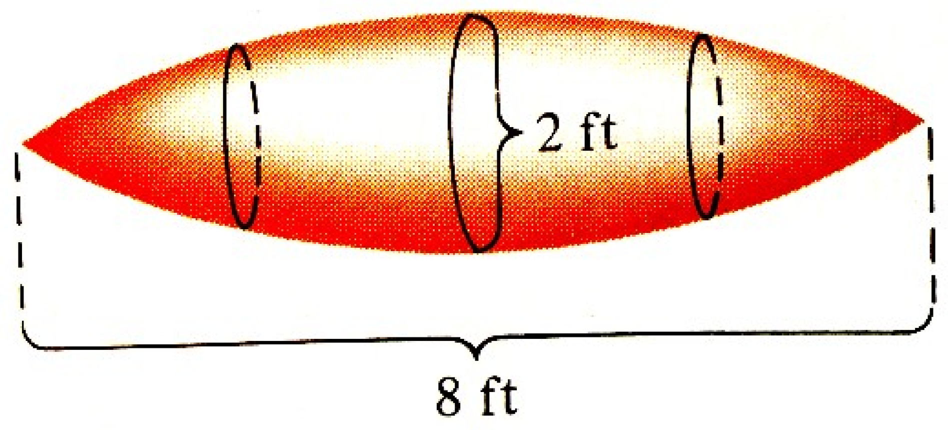


FIGURE 7.37

EXAMPLE 4 Disc method preferable

A pontoon is to be made in the shape shown in Figure 7.37. The pontoon is designed by rotating the graph of

$$y = 1 - \frac{x^2}{16}, \quad -4 \le x \le 4$$

about the x-axis, where x and y are measured in feet. Find the volume of the pontoon.

Solution: In Figure 7.38 we compare the disc and shell methods.

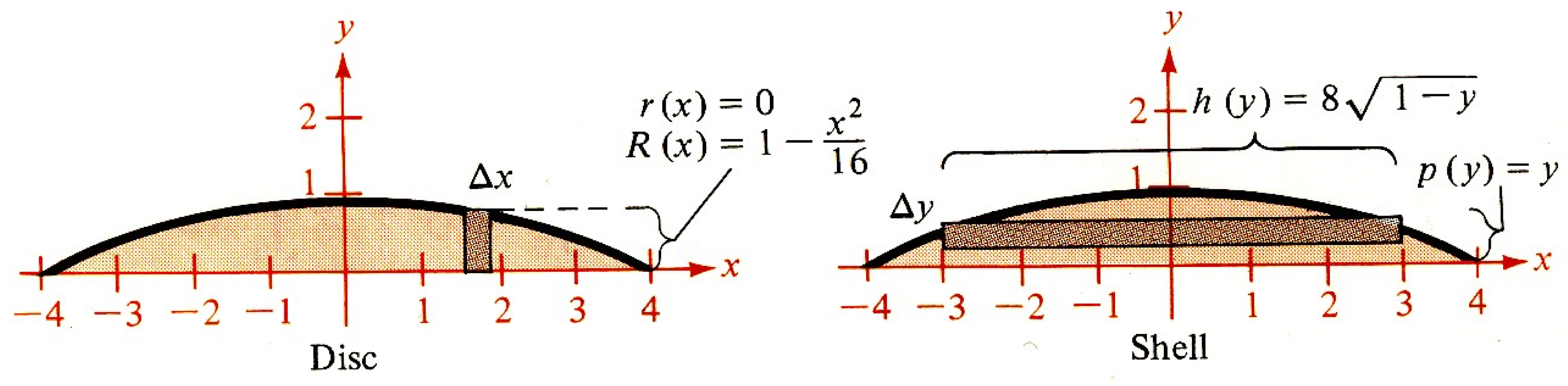


FIGURE 7.38

Using the shell method, we see that Δy is the width of the rectangle, $p(y) = y$, and $h(y) = 2x$. Solving for x in the equation $y = 1 - (x^2/16)$, we obtain

$$x = 4\sqrt{1 - y} \qquad \text{or} \qquad h(y) = 2x = 8\sqrt{1 - y}$$

Hence, the shell method results in the integral

$$V = 2\pi \int_0^1 8y\sqrt{1 - y}\, dy \qquad \text{Shell method}$$

Although this is not a particularly difficult integral, its evaluation does require a u-substitution. By contrast, the disc method yields the relatively simple integral

$$V = \pi \int_{-4}^{4} \left(1 - \frac{x^2}{16}\right)^2 dx = \pi \int_{-4}^{4} \left(1 - \frac{x^2}{8} + \frac{x^4}{256}\right) dx \qquad \text{Disc method}$$

$$= \pi\left[x - \frac{x^3}{24} + \frac{x^5}{1280}\right]_{-4}^{4}$$

$$= \frac{64\pi}{15} \approx 13.4 \text{ ft}^3$$

Note that for the shell method in Example 4 we had to solve for x in terms of y in the equation $y = 1 - (x^2/16)$. Sometimes solving for x is very difficult (or even impossible). In such cases we must use a vertical rectangle (of width Δx), thus making x the variable of integration. The position (horizontal or vertical) of the axis of revolution then determines the method to be used. This is illustrated in Example 5.

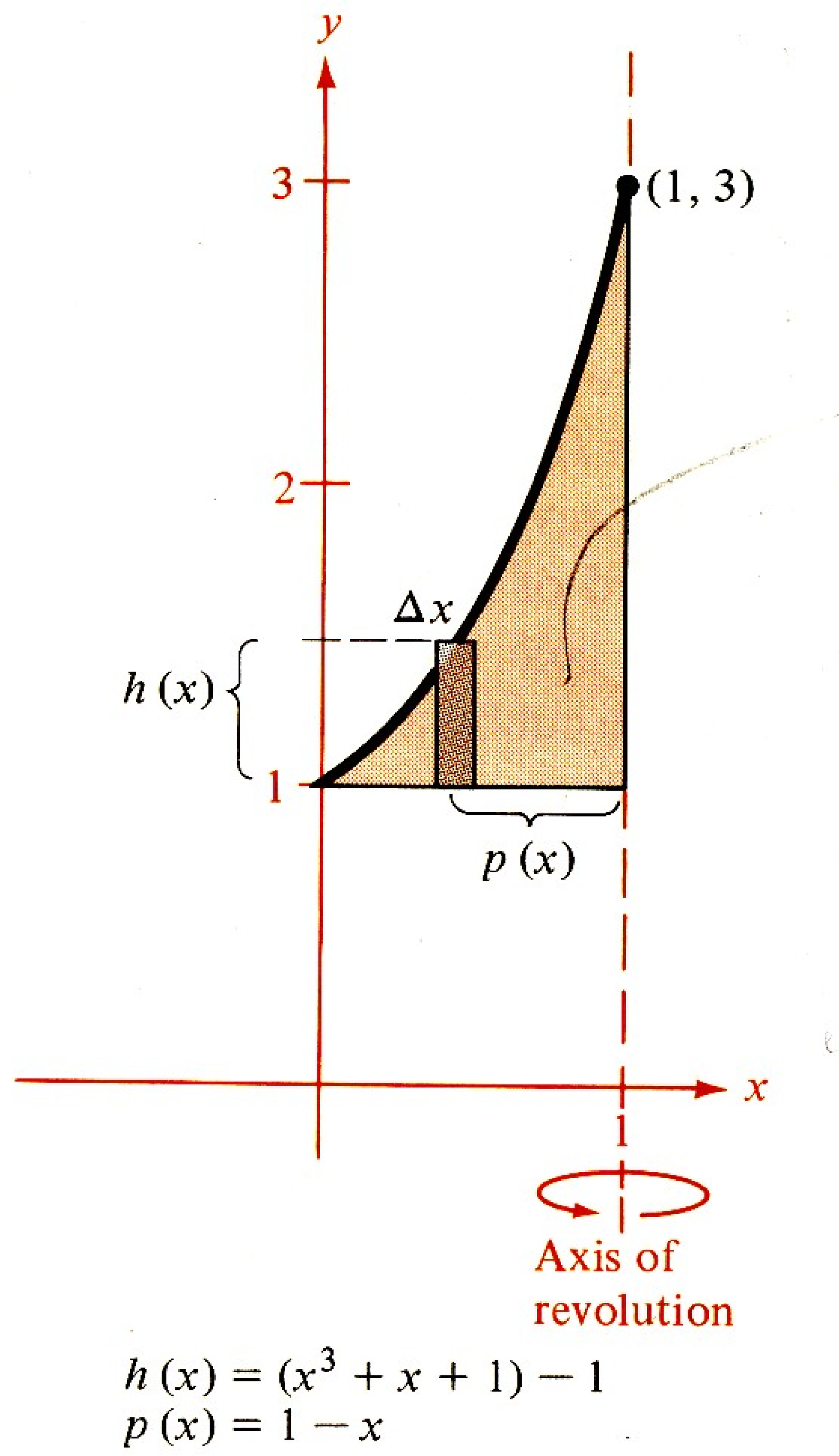


FIGURE 7.39

EXAMPLE 5 Shell method necessary

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^3 + x + 1$, $y = 1$, and $x = 1$ about the line $x = 1$, as shown in Figure 7.39.

Solution: In the equation $y = x^3 + x + 1$, we cannot easily solve for x in terms of y. (See the discussion at the end of Section 4.9.) Therefore, the variable of integration must be x, and we choose a vertical representative rectangle. Now, since the rectangle is parallel to the axis of revolution, we use the shell method, and the volume is

$$V = 2\pi \int_a^b p(x)h(x)\,dx = 2\pi \int_0^1 (1 - x)(x^3 + x + 1 - 1)\,dx$$

$$= 2\pi \int_0^1 (-x^4 + x^3 - x^2 + x)\,dx$$

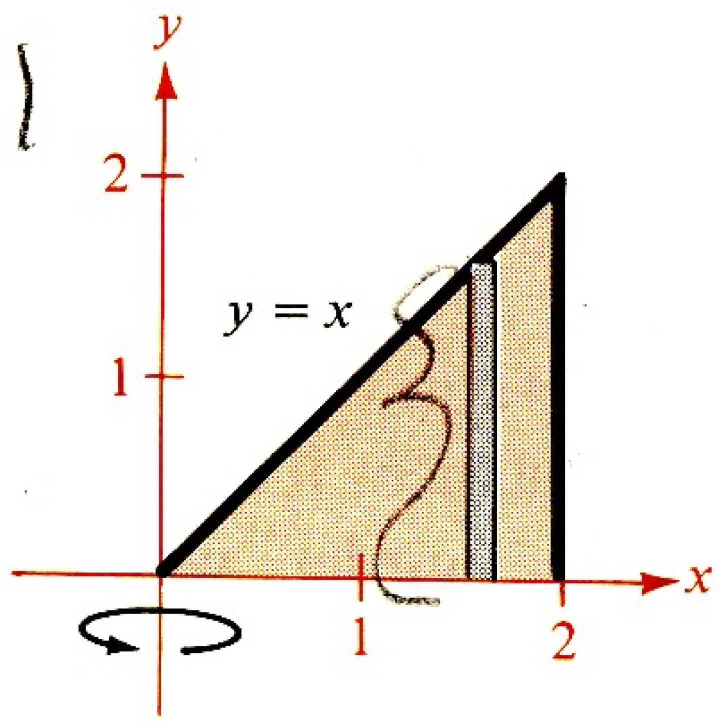
$$= 2\pi\left[-\frac{x^5}{5} + \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2}\right]_0^1$$

$$= 2\pi\left(-\frac{1}{5} + \frac{1}{4} - \frac{1}{3} + \frac{1}{2}\right) = \frac{13\pi}{30}$$

Section Exercises 7.3

In Exercises 1–20, use the shell method to find the volume of the solid generated by revolving the given plane region about the indicated line.

1.



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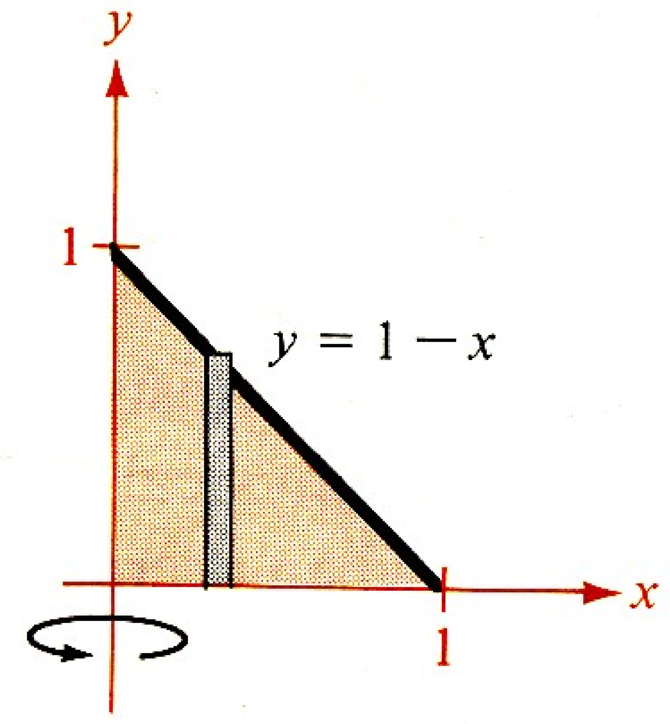
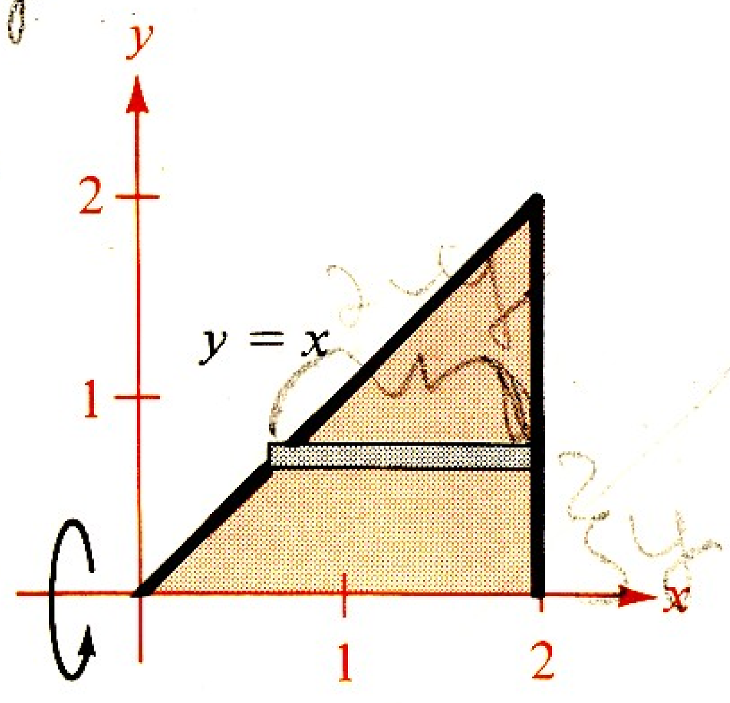
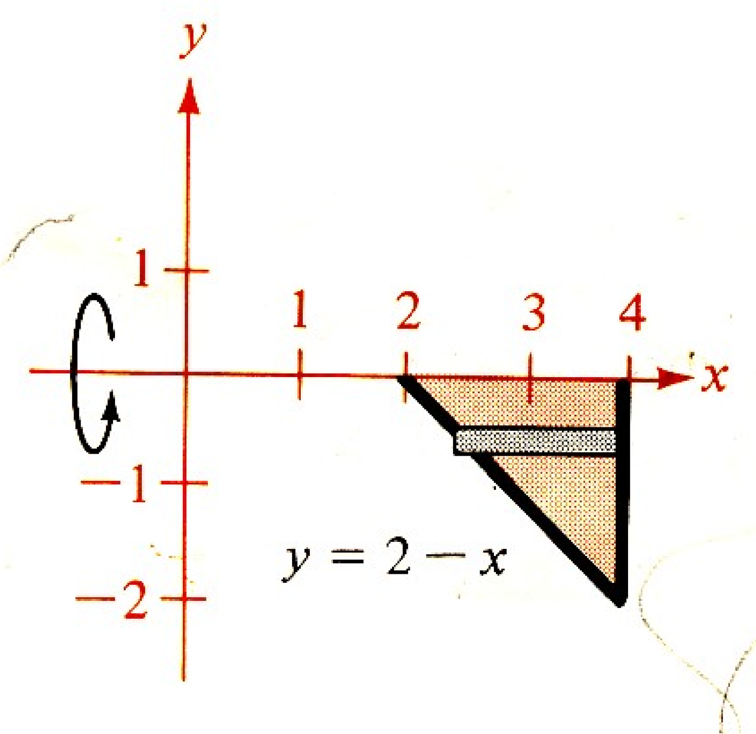


3.



4.



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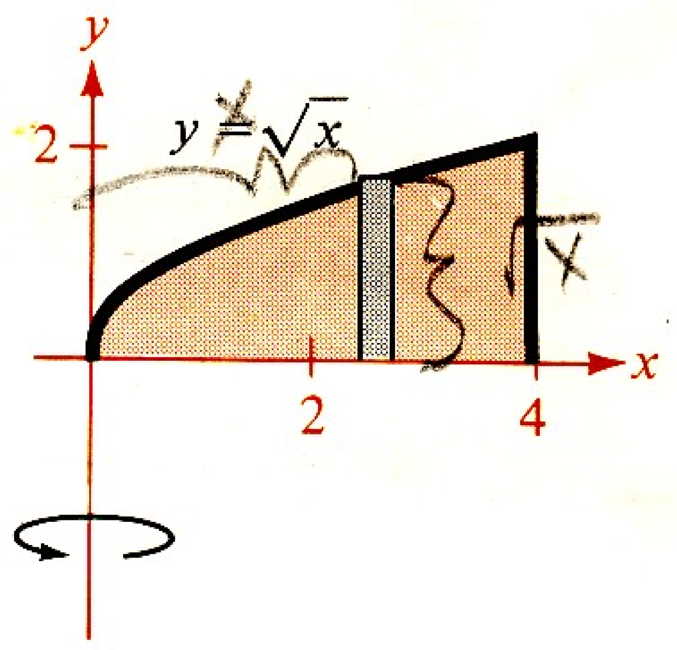
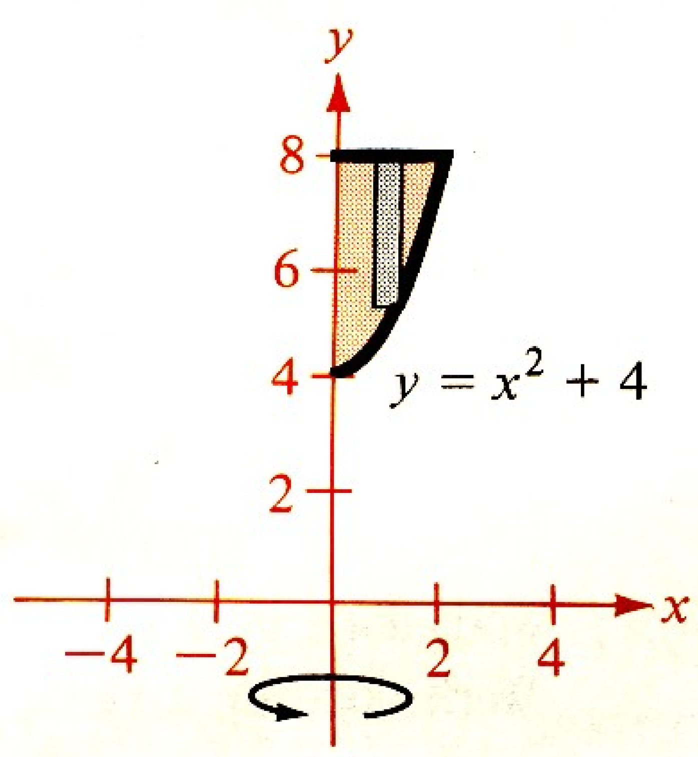


6.



7. $y = x^2$, $y = 0$, $x = 2$, about the y-axis

8. $y = x^2$, $y = 0$, $x = 4$, about the y-axis

9. $y = x^2$, $y = 4x - x^2$, about the y-axis

10. $y = x^2$, $y = 4x - x^2$, about the line $x = 2$

11. $y = x^2$, $y = 4x - x^2$, about the line $x = 4$

12. $y = \dfrac{1}{x}$, $x = 1$, $x = 2$, $y = 0$, about the x-axis

13. $y = 4x - x^2$, $y = 0$, about the line $x = 5$

14. $x + y^2 = 9$, $x = 0$, about the x-axis

15. $y = 4x - x^2$, $x = 0$, $y = 4$, about the y-axis

16. $y = 4 - x^2$, $y = 0$, about the y-axis

17. $y = \sqrt{x}$, $y = 0$, $x = 4$, about the line $x = 6$